XVII. *On the Geometrical Forms of Turbinated and Discoid Shells.*

By the Rev. H. Moseley, *M.A., of St. John's College, Cambridge, Professor of Natural Philosophy and Astronomy in King's College. Communicated by* Thomas Bell, *Esq. F.R.S.*

Received June 14,—Read June 21, 1838.

THE surface of any turbinated or discoid shell may be imagined to be generated by the revolution about a fixed axis (the axis of the shell) of the perimeter of a geometrical figure, which, remaining always geometrically similar to itself, increases continually its dimensions.

In discoid shells the generating figure retains its position upon the axis as it thus revolves, as in the *Nautilus Pompilius* (Plate IX. fig. 3.), and the *Argonaut*. In turbinated shells, including the great families of Trochi, Turbines*, Murices and Strombi, it slides continually along the axis of its revolution (fig. 4.). In some great classes of shells, as the Ammonites, the *Nautilus scrobiculatus*, the *Nautilus spirula*, the *Helix cornea*, the *Trochus perspectivus*, the Nerita, the generating figure increases its distance from the axis at the same time that it increases its dimensions and revolves.

Among the generating figures of conchoidal surfaces are to be found various known geometrical forms. The generating figure of the *Conus Virgo* is a triangle, that of the *Trochus telescopicus* and of the *Trochus Archimedis*, a trapezoid. The species of the genus Turbo have for their generating figure a curve, of double curvature, of a circular or elliptic form, to whose perimeter the axis of revolution is a tangent. The *Nautilus Pompilius* is generated by the revolution about its shorter diameter of a plane curve, approaching very nearly to a semi-ellipse (fig. 3.); and the Cypræa by the revolution of a similar curve about its longer diameter.

There is a mechanical uniformity observable in the description of shells of the same species, which at once suggests the probability that the generating figure of each increases, and that the spiral chamber of each expands itself, according to some simple geometrical law common to all. To the determination of this law, if any such exist, the operculum lends itself, in certain classes of shells, with remarkable facility. Continually enlarged by the animal, as the construction of its shell advances, so as to fill up its mouth, the operculum measures the progressive widening of the spiral chamber, by the progressive stages of its growth.

Of these progressive stages of the growth of the operculum, distinct traces remain

* The beautiful shell *Turbo scalaris* (Ventletrap) may be taken as an easy illustration of the properties to be described in this paper.

on its surface, under the form in the Turbines (fig. 1.) of certain *curved* lines, and in the Neritæ (fig. 2.) of certain *straight* lines, passing from the margin of the operculum (which if produced they would intersect) to a certain spiral line marked deeply upon its face. To this spiral they are tangents, and may be supposed to generate it by their consecutive intersections. The spiral eventually passes into the margin of the operculum, and for a considerable distance traces it.

If the eye be made to traverse one of the curved lines first spoken of in the operculum of the Turbo, or one of the straight lines in the Nerita, from its margin to the point where it loses itself in the spiral, and if it then follow the spiral until it returns to the point in the margin whence it set out, it will have traversed the boundary of a figure which was once the actual boundary of the operculum, which therefore indicates one stage of its growth, and of which all, similarly traced, will be seen to have similar geometrical forms.

It will further be apparent from this examination, that the operculum has increased at each stage of its growth, not throughout its whole margin at once, but on a series of different portions of it lying in different consecutive positions round it; each such addition being so made as to preserve the above-mentioned geometrical similarity of the whole*. In all the similar geometrical figures thus visible upon the face of the operculum, and which have in succession constituted its limits, the pole of the spiral will moreover be seen to occupy a similar position. The linear dimensions of any two of them ($P_1 C Q_1$ and $P_2 C Q_2$) are then to one another as the radii vectores drawn to similar points in them, and therefore as those ($P P_1$ and $P P_2$) drawn to the extremities of the boundary by which they unite.

To determine, therefore, the law according to which the linear increase of the operculum takes place, that is, the law according to which the linear increase of the section of the chamber of the shell takes place, we have only to determine the law according to which the radii vectores, drawn to successive points of the spiral visible upon the operculum, increase, that is, we have only, geometrically, to determine the spiral.

Now in every case this spiral is the logarithmic spiral.

A slight inspection of it is sufficient to suggest the probability that the angle at which it intersects its radius vector is everywhere the same, and this supposition is fully confirmed by direct admeasurements grounded upon the following property of the logarithmic spiral, "That the distances of successive spires, measured upon the same radius vector produced, from the pole and from one another, are respectively in geometrical progression; the common ratio of the progression being in both cases $\epsilon^{2\pi \cot A}$, where A is the constant angle of the spiral†."

* The whole class of shells Haliotis affects the method of formation here described. The *shell* itself is in this class generated by additions upon one margin, as in other classes the *operculum* is generated.

† Let R_n, R_{n+1}, R_{n+2} be consecutive radii vectores taken as above, and R_0 the radius vector corresponding to $\theta = 0$ $\therefore R_n = R_0 \epsilon^{\theta \cot A}$, $R_{n+1} = R_0 \epsilon^{(\theta + 2\pi) \cot A}$, $R_{n+2} = R_0 \epsilon^{(\theta + 4\pi) \cot A}$ $\therefore R_{n+1} = \epsilon^{2\pi \cot A}$. R_n and $R_{n+2} - R_{n+1} = \epsilon^{2\pi \cot A} (R_{n+1} - R_n)$.

The following distances were measured upon three different opercula from the poles of their spiral curves to their successive whorls; the distances in the same column being measured on the same radius vector produced. It will be perceived that for the same operculum these distances have the same ratio consecutively to one another; the deviation from this law in no case exceeding that error which of necessity attaches to the method of admeasurement.

Operculum of the Order Turbo, No. I.

Distance in inches.	Ratio.	Distance in inches.	Ratio.	Distance in inches.	Ratio.	Distance in inches.	Ratio.
·24 ·55 1·28	2·28 2·32	·16 ·37 ·85	2·31 2·3	·2 ·6 1·38	2·3 2·3	·18 ·42 ·94	2·3 2·24

Operculum, No. II.

Distance in inches.	Ratio.	Distance in inches.	Ratio.	Distance in inches.	Ratio.
·32 1·25	3·9	·25 1·04	4	·155 ·62	3·87

Operculum, No. III.

Distance in inches.	Ratio.	Distance in inches.	Ratio.
·6 ·8 ·99 1·27	1·33 1·23 1·28	·74 ·95 1·215	1·28 1·28

The spiral of the operculum is then a logarithmic spiral. Now its linear dimensions in the different successive stages of its progress have been shown to be as the successive radii vectores of its spiral. The increments of its linear dimensions are then as the increments of these radii vectores. But by a fundamental property of the logarithmic spiral, the increments of its radii vectores, corresponding to equal increments in their angles of revolution, are as the radii vectores themselves. Thus, then, it follows that the increments of the linear dimensions of the operculum, corresponding to equal angular distances round its pole, are as its existing linear dimensions; and, therefore, that the increments of the linear dimensions of the section of the spiral chamber corresponding to these are everywhere as its existing linear dimensions.

The animal, as he advances in the construction of his shell, increases continually his operculum, so as to adjust it to its mouth.

He increases it, however, not by additions made at the same time all round its margin, but by additions made only on one side of it at once. One edge of the operculum thus remains unaltered as it is advanced into each new position, and placed in a newly formed section of the chamber similar to the last, but greater than it.

That the same edge which fitted a portion of the first less section should be capable of adjustment, so as to fit a portion of the next similar but greater section, supposes a geometrical provision in the curved form of the chamber of great apparent complication and difficulty. But God hath bestowed upon this humble architect the practical skill of a learned geometrician, and he makes this provision with admirable precision in that curvature of the logarithmic spiral which he gives to the section of the shell. This curvature obtaining, he has only to turn his operculum slightly round in its own plane as he advances it into each newly formed portion of his chamber, to adapt one margin of it to a new and larger surface and a different curvature, leaving the space to be filled up by increasing the operculum wholly on the other margin.

To make this apparent, let the following be received as a characteristic property of the logarithmic spiral: "That lines anywhere drawn from its pole, inclined to one another at the same angle, will intercept between them branches of the curve which, however different their linear dimensions, will be geometrically similar to each other." So that if two lines given in position be imagined to be drawn from the pole of such a spiral, parallel to its plane, and the spiral be then imagined to be put in motion in its own plane round its pole, then as its curve revolved under these lines they would intercept portions of it continually increasing, or continually diminishing in dimensions, and continually receding from or approaching the pole, but all geometrically similarly to each other and similarly placed."

Now each new section of the chamber of the shell being similar to the preceding section, but greater than it, if the operculum were thrust forward into this greater section without being turned round in its own plane, any portion of its edge would manifestly present to the corresponding portion of the perimeter of the new section a similar but a less curve, which could not be made to coincide with it. If, however, the operculum be imagined to be turned round in its own plane about its pole in the opposite direction to that in which the spiral increases, the curve presented by it to this portion of the perimeter of the section will continually approach it, increasing its dimensions, but remaining similar to it, so that at length it will coincide with it. And thus one margin of the operculum will be made everywhere to fit itself to the side of the chamber, the coincidence of the other margin remaining to be produced by new matter added to it.

It will be apparent from a simple inspection of the operculum that the animal does thus turn it round in its own plane as he advances it, with what is called a screw motion.

Such is the theory of the growth of the operculum. There is traced in it the application of properties of a geometric curve to a mechanical purpose by Him who metes the dimensions of space and stretches out the forms of matter according to the rules of a perfect geometry,—properties which, like so many others in nature, may have also their application in art. It instructs us how to shape a tube of a variable section, so that a piston driven along it shall, by one side of its margin, coincide continually with its surface as it advances, provided only the piston be made at the same time continually to revolve in its own plane.

The investigation has now arrived at a point from which the law of the geometrical description of turbinated shells can be enunciated with greater precision. "They are generated by the revolution about a fixed axis (the axis of the shell) of a curve, which continually varies its dimensions according to the law, that each linear increment, corresponding to a given angular increment, shall vary as the existing dimensions of the line of which it is the increment (the law of the description of the logarithmic spiral), and which curve either retains its position upon the axis, or moves along it with a motion of translation in the direction of its length."

This law is readily subjected to verification by admeasurement.

It is clear that, if it obtain, similar linear dimensions measured at similar points of successive whorls, should be in geometric progression. Thus if the generating curve (as in the *Nautilus Pompilius*) revolve about the axis without at the same time sliding along it, and a section be made through the centre of the shell perpendicular to the axis, then will the section be (if this law be true) a spiral curve, whose distances from the axis, measured on the same radius vector, are in geometrical progression, and which is therefore a logarithmic spiral.

In the more general case, in which the generating curve, as in the *Turbo scalaris*, slides forwards upon the axis as it revolves, increasing at the same time its linear dimensions according to the law of the logarithmic spiral, it is clear that the surfaces of the successive whorls would interfere with one another, and that thus the uniformity of the spiral chamber would be destroyed, unless the motion of translation (or the sliding motion) of the curve, by which the space allowed to each whorl upon the axis is determined, were governed by some law corresponding to that which governs the linear dimensions of the whorl; unless, in short, the spaces allowed to the widths of successive whorls upon the axis varied in the same progression as the widths themselves vary. A similar principle applies to the distances of the whorls measured upon the surface of the shell in the same plane passing through the axis. These distances are, in fact, in this case, similar linear dimensions of successive whorls, and are therefore subject, according to the theory, to the law of the logarithmic spiral, and like the distances of successive whorls of that spiral, on the same radius vector, are in geometric progression.

Nautilus Pompilius.

These conclusions were directly verified by the following observations. A shell of the *Nautilus Pompilius* was cut through the middle in a direction perpendicular to its axis, and a tracing was taken of the section of its spiral surface; this tracing is copied in fig. 6.

It was made from the dark line which shows, on the section of the internal whorls of the shell, the line of that pearly surface which the animal deposits as a covering to its completed portion, as it advances in the construction of it. It is important to make this observation, because as it extends one whorl of its shell over another, the animal deposits continually upon the pearly surface of this last a new coating of shell, and thickens it; and it is in the centre of this thickened section that is to be found that section of the pearly surface, of which the edge of the external whorl is a continuation, and from which this tracing was taken.

It will be found that the distance of any two of its whorls measured upon a radius vector is *one-third* that of the two next whorls measured upon the same radius vector. Thus

$a\,b$ is one-third of $b\,c$,
$d\,e$ is one-third of $e\,f$,
$g\,h$ is one-third of $h\,i$,
$l\,k$ is one-third of $l\,m$.

The curve is therefore a *logarithmic spiral.*

Turbo duplicatus.

From the apex of a large specimen of the *Turbo duplicatus* a line was drawn across its whorls, and their widths were measured upon it in succession, beginning from the last but one. The measurements were, as before, made with a fine pair of compasses and a diagonal scale. The sight was assisted by a magnifying glass.

In a parallel column to the admeasurements are the terms of a geometric progression, whose first term is the width of the widest whorl measured, and whose common ratio is 1·1804.

Widths of successive whorls measured in *inches* and parts of an *inch*.	Terms of a geometrical progression, whose first term is the width of the widest whorl, and whose common ratio is 1·1804.
1·31	1·31
1·12	1·1098
·94	·94018
·8	·79651
·67	·67476
·57	·57164
·48	·48427
·41	·41026

Yet further to verify this remarkable coincidence of the widths of successive whorls with the mathematical law of a geometric progression, the following property of such a progression was determined: "that μ representing the ratio of the sum of every even number (m) of its terms to the sum of half that number of terms, the common ratio (r) of the series is represented by the formula

$$r = (\mu - 1)^{\frac{2}{m}}."$$

The following measurements were then made, beginning from the second and third whorls respectively:

Width of six whorls in inches.	Width of three whorls in inches.	Ratio μ.
5·37	2·03	2·645
4·55	1·72	2·645
Width of four whorls in inches.	**Width of two whorls in inches.**	**Ratio μ.**
4·15	1·74	2·385
3·52	1·47	2·394

By the two first admeasurements the formula gives

$$r = (1{\cdot}645)^{\frac{1}{3}} = 1{\cdot}1804.$$

By the mean of the ratios deduced from the two second admeasurements it gives

$$r = (1{\cdot}389)^{\frac{1}{2}} = 1{\cdot}1806.$$

It is scarcely possible to imagine a more accurate verification than is deduced from these larger admeasurements, and we may with safety annex to the species *Turbo duplicatus* the characteristic number 1·18.

Buccinum subulatum.

A line was drawn from the apex of this shell across its whorls as in the last, and the following admeasurements were similarly made.

Widths of successive whorls by admeasurement in inches.	Terms of a geometrical progression, whose first term is the width of the widest whorl and ratio, 1·13.
1·14	1·14
1·00	1·0089
·9	·89279
·79	·79008
·7	·69919
·62	·61875
·54	·54757

In verification of the above the following larger admeasurements were made, beginning respectively from the last whorl, the last but one, and the last but two.

Width of six whorls.	Width of three whorls.	Ratio μ.
5·10	2·08	2·45
4·52	1·84	2·45
3·94	1·60	2·45

From these admeasurements we have, by the formula $r = (\mu - 1)^{\frac{2}{m}}$,

$$r = (1{\cdot}45)^{\frac{1}{3}} = 1{\cdot}1318.$$

Turbo phasianus.

Three lines were drawn from the apex of this shell in different directions across its whorls, and the following admeasurements were made upon them:

Widths of successive whorls in inches by admeasurement.	Terms of a geometrical progression, whose first term is the width of the widest whorl and ratio, 1·75.
First line.	First line.
2·55	2·55
·44	·44
·25	·251
Second line.	Second line,
·98	·98
·56	·56
·32	·323
Third line.	Third line.
·7	·7
·4	·4
·23	·228

The remarkable accordance of the measured with the calculated widths of the whorls in this shell is to be attributed to the precision with which the line of separation of the whorls is traced upon it. A great number of admeasurements were similarly made upon other shells of the genera Trochus, Strombus and Murex; some of them were cut through the axis longitudinally; and similar measurements were made by drawing lines from the apex across the section. From all these the same result was obtained.

Thus to each particular species of shell is annexed a characteristic number, being the ratio of the geometric progression of similar successive linear dimensions of its whorls; from which number is deducible the constant angle of the particular logarithmic spiral which is affected by that species of shell (see equation 26. in the following mathematical discussion). This number, or this angle, connected as it is of necessity with the circumstances of the animal's growth and the manner of its existence, is determinable by actual admeasurement, and may be available for the purposes of classification; it may suggest relations to which the observations of naturalists may usefully be directed, and eventually become linked with characteristic forms and modes of molluscous existence*.

* The whole race of land shells, for instance, will certainly (from the nature of the case) be distinguished from the aquatic shells by a wide difference in the numbers characteristic of the species of the two groups.

Why the Mollusks who inhabit turbinated and discoid shells should, in the progressive increase of their spiral dwellings, affect the particular law of the logarithmic spiral, is easily to be understood. Providence has subjected the instinct which shapes out each, to a rigid *uniformity* of operation.

This uniformity manifests itself in turbinated shells in respect to their axes. Now the law of the logarithmic spiral, considered under its more general form of a curve of double curvature, is the only one according to which the Mollusk can wind its spiral dwelling in an uniform direction through the space round its axis, in respect to that axis. Under this general form it may be geometrically defined as the curve whose tangent retains always the same angular position in respect to its axis*, and in respect to a line drawn from the point where it touches the curve perpendicular to the axis; or in other words, which traverses the space round the axis always in the same direction in respect to it.

A second property of the logarithmic spiral, equally referring itself to the uniformity of the animal's operations about the axis of its shell, is this; that it has everywhere the same geometrical curvature, and is the only curve except the circle† which possesses this property.

Certain physiological facts having reference to the growth of the Mollusk are deducible from the geometrical description of its shell. If it be a *land* shell, its capacity may be supposed (reasoning from that principle of economy which is an observable law in Nature) to be precisely sufficient for the reception of the animal who built it. If it be an *aquatic* shell, it serves the animal at once as a habitation and as a float; enabling it to vary its buoyancy according as it leaves a greater or a less portion of the narrower extremity of its chamber unoccupied, and thus to ascend or descend in the water, at will. Now that its buoyancy, and therefore the facility of thus varying its position, may remain the same at every period of its growth, it is necessary that the increment of the capacity of its float should bear a constant ratio to the corresponding increment of its body, a ratio which always assigns a greater amount to the increment of the capacity of the shell than to the corresponding increments of the animal's bulk. Thus the chamber of the *aquatic* shell is increased, not only, as is the land shell, so that it may contain the greater bulk of the Mollusk, but so that more and more of it may be left unoccupied. Now the capacity of the shell and the dimensions of the animal began together, and they increase thus in a constant ratio; the whole bulk of the animal bears therefore a constant ratio, of greater inequality, to the whole capacity of the shell, in *aquatic* shells: in *land* shells, it is probably equal to it.

Now let the generating curve of a shell be conceived to describe, as it revolves round its axis, a series of successive equal angles, represented each by $\Delta\,\Theta$. Corresponding to these equal increments of the angle of revolution of the generating

* So that moved parallel to itself until it intersected the axis, it would always intersect it at the same angle.

† The circle may, in fact, be considered a logarithmic spiral, the constant inclination of whose tangent to its radius vector is a right angle. Of all curves, this spiral, considered as thus including the circle, is the simplest.

curve, will be certain increments of the capacity of the shell; and it appears from the following mathematical investigation of the properties of conchoidal surfaces (see equation 19.), that the increments of the capacity of the shell, thus taken, will be in a constant ratio to the then existing whole capacities of the shell. The increment of the animal's bulk corresponding to each of these increments of the shell must then be in a constant ratio to its then existing bulk; that is, the animal's growth corresponding to a given increment, $\Delta \Theta$, in the angle described by the generating curve of its shell, is proportional always to its existing growth.

Let us now suppose that the physical living energies of the animal (those by which it grows), at any time, are proportional to its then existing growth; and therefore that its growth in any increment of time is proportional to its growth up to that time (a supposition which possesses an independent probability). From the conclusion before arrived at, and from this supposition, it follows that the growth of the animal corresponding to a given increment, $\Delta \Theta$, in the angle of revolution of the generating curve, and the growth corresponding to a given increment of time, are each proportional to the animal's whole then existing growth, and therefore to one another; and, since they begin together, that the whole angle, Θ, of revolution of the generating curve of the shell, is proportional to the whole corresponding time of the animal's growth, and therefore that the whole number of whorls, and parts of whorls, is proportional to its whole age: a conclusion which, like the supposition whence it is deduced, possesses an independent probability.

The separate probability of each of the two suppositions, "that the physical energies of the Mollusk, as developed in its growth in a given increment of time, are proportional to its whole then existing growth*, and that its age is always proportional to the whole angle which, in the construction of the shell, it has then described round its axis," is greatly increased by the necessary relation which is here shown to obtain between them; a relation, by reason of which, either supposition being made, the other becomes a conclusion.

The form of the Mollusk being supposed to remain geometrically similar to itself, the surface of its mantle, by which organ it deposits its shell, of necessity varies as the square of its linear dimensions, whilst the whole bulk of the animal varies as the cube of its linear dimensions. But, as its whole bulk, varies its active living and growing energy, and therefore the amount of the deposition of its shell in a given time; this last, then, varies as the cube of its linear dimensions; but the surface of the depositing organ (the mantle) varies only as the square of the same linear dimensions. Besides, then, the organic increase of the surface of the mantle, there must be an increased functional activity of all its organs, varying as its simple linear dimensions.

This increased functional activity of the surface of the depositing organ, varying

* May not this law of the growth of a Mollusk have its analogy in other forms of animal life, and perhaps in vegetable life?

simply as the linear dimensions of the animal or its shell, offers an analogy, and has perhaps a relation, to the increase of the section of the shell, according to the same law of its simple linear dimensions.

Subjoined to this paper is a mathematical discussion of the following geometrical and mechanical elements of a conchoidal surface: its VOLUME, the dimensions of its SURFACE, the CENTRE OF GRAVITY of its contained solid, the CENTRE OF GRAVITY of its surface.

These elements are determined (the law of the logarithmic spiral being supposed) by certain transcendental functions, having constant factors dependent for their amount upon the statical moments and the moments of inertia of the generating figures and of their areas.

The object proposed in the determination of these elements was their application to a discussion of the hydraulic theory of shells; yet further, if possible, to develope that wisdom of God which shaped them out and moulded them; and especially in reference to the particular value of the constant angle which the spiral of each species of shell affects,—a value connected by a necessary relation with the economy of the material of each, and with its stability, and the conditions of its buoyancy*.

The paper concludes with a discussion of the general EQUATIONS to a conchoidal surface in respect to systems of polar and of rectangular coordinates.

To determine the Volume of a Conchoidal Solid.

Suppose the generating curve to be a plane curve, and let it (first) retain its position upon the axis as it revolves, varying its dimensions.

Let P C and Q C (fig. 3.) be two of its positions, inclined at the angle $\Delta\,\Theta$, and including between them the elementary solid P C Q.

Imagine the plane P C to have revolved about A z through the angle $\Delta\,\Theta$ without altering its dimensions, the solid generated by it would then, by the theorem of GULDINUS, be represented by $M \,.\, \Delta\,\Theta$, where M represents the statical moment of the plane P C about the axis A z.

The elementary solid imagined to be in like manner generated by the revolution of Q C through the angle $\Delta\,\Theta$, will similarly be represented by $(M + \Delta\,M)\,\Delta\,\Theta$.

Now between these two imaginary solids is evidently the actual elementary solid P C Q. Calling then V the volume of the solid to be determined, we have

$$M\,\Delta\,\Theta < \Delta\,V < (M + \Delta\,M)\,\Delta\,\Theta.$$

Or, considering M and V as functions of Θ, and expanding by TAYLOR's theorem,

$$M\,\Delta\,\Theta < \frac{d\,V}{d\,\Theta}\,\Delta\,\Theta + \frac{d^2\,V}{d\,\Theta^2}\cdot\frac{(\Delta\,\Theta)^2}{1\,.\,2} + \&c. < M\,\Delta\,\Theta + \frac{d\,M}{d\,\Theta}\,(\Delta\,\Theta)^2 + \&c.$$

* As illustrative of this remark, it may here be mentioned that the shell of the *Nautilus Pompilius* has, hydrostatically, an *A*-statical surface. If placed with any portion of its surface upon the water, it will immediately turn over towards its smaller end, and rest only on its mouth. Those conversant with the theory of floating bodies will recognise in this an interesting property.

And this is true for all values of Θ:

$$\therefore \frac{dV}{d\Theta} = M,$$

and

$$V = \int M \, d\Theta. \quad \ldots \ldots \ldots \ldots \quad (1.)$$

If we imagine the plane C Q to slide along the axis A z (fig. 4.) without otherwise altering its position, the elementary solid included between it and P C will retain the same volume as it had before; for the two planes P C and Q C may be divided into the same number of similar elements, whose corresponding angles being joined, the solid element included between them will be divided into as many pyramidal frusta, the volume of each of which will remain unaltered by the supposed displacement of C Q, since each such frustum may be imagined to be made up of two pyramids, the base of each of which will remain the same after the displacement, and their bases and vertices between the same parallels. Thus, then, the volume determined by the above formula is that of the conchoidal solid under its most general form.

To determine the Area of a Conchoidal Surface.

Let U represent the whole area of the surface (fig. 3.), and Δ U the elementary area intercepted between the positions P C and Q C of the generating curve, supposed to revolve without otherwise altering its position on the axis.

Take Δ S to represent the element P Q of the curve described by the extremity P of the revolving axis P C of the generating curve.

Imagine the generating curve to describe, without altering its dimensions, an angle about the axis A z, such that the circular arc, described on this supposition by the point P, may equal P Q or Δ S. This angle will be represented by $\frac{\Delta S}{R}$.

The generating curve remaining always similar to itself, its statical moment about A z is a function of Θ or of R. Let it be represented by N, and considered a function of R. The elementary surface which the curve C P will generate, on the supposition just made, will then be represented by $\frac{\Delta S}{R} \cdot N$, according to the property of GULDINUS.

A surface similarly generated by C Q will in like manner be represented by

$$\frac{\Delta S}{R + \Delta R} (N + \Delta N).$$

Now the dimensions of the actual element of the conchoidal surface lie between the dimensions of these two imaginary surfaces.

This will be seen if we conceive any number of planes passing through the axis C D, at right angles to A z, to intersect all three of the surfaces spoken of. The intercepted parts will be strips of the three surfaces, all of the same length, but of breadths, of which those of the surface described by P C will be the least, and those described by Q C the greatest.

$$\therefore \frac{N}{R} \Delta S < \Delta U < \frac{N + \Delta N}{R + \Delta R} \cdot \Delta S.$$

Considering, therefore, S, U, and N as functions of R, expanding by TAYLOR's theorem and dividing by Δ R

$$\frac{N}{R} \cdot \frac{dS}{dR} + \frac{d^2S}{dR^2} \cdot \frac{N}{R} \cdot \frac{\Delta R}{1.2} + \&c. < \frac{dU}{dR} + \frac{d^2U}{dR^2} \cdot \frac{\Delta R}{1.2} + \&c. < \frac{N}{R} \cdot \frac{dS}{dR}$$

$$+ \left\{ \frac{dS}{dR} \left(\frac{1}{N} \frac{dN}{dR} - \frac{1}{R} \right) + \frac{1}{2} \frac{d^2S}{dR^2} \right\} \Delta R + \&c.$$

The second of these series having, for all values of Δ R, a value intermediate between the other two, and the first terms of these other two being equal; the first terms of the three series are equal.

$$\therefore \frac{dU}{dR} = \frac{N}{R} \cdot \frac{dS}{dR}$$

and

$$U = \int \frac{N}{R} \cdot \frac{dS}{dR} \cdot dR \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2.)$$

which to adapt it for integration, (R being a function of Θ) may be put under the form

$$U = \int \frac{N}{R} \cdot \frac{dS}{dR} \cdot \frac{dR}{d\Theta} \cdot d\Theta.$$

The expression for the area of the surface thus determined, on the supposition that the generating curve does not alter its position in respect to the axis otherwise than by revolving round it, is the same with that of the surface which would be generated by a curve which, as it revolved about the axis, slided along it, a different form being assigned to the function N. For if we imagine a conchoidal surface of this general form (fig. 4.) to be intersected by planes, exceedingly near to one another, passing through its axis, and at the same time to be traversed, as the surfaces of turbinated shells usually are, by spiral lines parallel to the direction of the whorl, and which may be understood to mark the paths of given points in the generating curve*; then each element of the surface intercepted between two of the planes spoken of will, by these spiral lines, be divided into a series of oblique parallelograms, two adjacent sides (containing an acute angle) of each of which, may be considered as intersections with the conchoidal surface of two planes, which intersect one another in an ordinate of the generating curve; one of these planes is a tangent to one of the spiral lines spoken of, and the other is the plane of the generating curve itself. Now let us suppose the inclination of these planes to one another to be *constant*, as is always the case in shells, and let it be represented by A. Let moreover the inclination, to its ordinate,

* This demonstration will be best understood by referring to the actual surface of a turbinated shell on which the spiral lines are visible.

of the tangent to the generating curve be represented by ϕ; and the inclination to the same ordinate of the tangent to the spiral line by σ. We have then given the inclination A of two planes to one another, and the inclinations ϕ and σ of two lines, drawn in them respectively, to the intersection of these planes; whence by a well-known formula of spherical trigonometry, if ι represent the inclination of these lines to one another,

$$\cos \iota = \cos \phi \cos \sigma + \sin \phi \sin \sigma \cos \mathrm{A}.$$

Moreover, if the two adjacent sides of the parallelogram, being elements of the generating curve, and the spiral, be represented by Δs and $\Delta \mathrm{S}$; then since their inclination is ι, the area of the parallelogram is represented by $\Delta \mathrm{S} . \Delta s \sin \iota$. Now let us suppose the generating curve to revolve, not altering its dimensions, but sliding along the axis; then

$$\sigma = \frac{\pi}{2} \therefore \cos \iota = \sin \phi \cos \mathrm{A}, \text{ and } \sin \iota = \sqrt{1 + \sin^2 \mathrm{A} \tan^2 \phi} . \cos \phi;$$

also in this case

$$\Delta \mathrm{S} = y \Delta \Theta \operatorname{cosec} \mathrm{A};$$

the area of the elementary parallelogram becomes then

$$y \sqrt{\operatorname{cosec}^2 \mathrm{A} + \tan^2 \phi} . \cos \phi . \Delta s . \Delta \Theta, \quad \text{or } y \sqrt{\operatorname{cosec}^2 \mathrm{A} + \frac{d x^2}{d y^2}} \cdot \frac{d y}{d s} \Delta s \Delta \Theta;$$

so that the whole surface of the elementary slice intercepted between two planes passing through the axis which are inclined to one another at an angle $\Delta \Theta$, is on this supposition,

$$\Delta \Theta \int y \sqrt{\operatorname{cosec}^2 \mathrm{A} + \frac{d x^2}{d y^2}} \cdot d y.$$

Suppose the integral in this expression to be represented by N^1, then N^1 will become N (as it ought) in that particular case in which, the curve not sliding along the axis, A becomes $\frac{\pi}{2}$.

Now we may reason in respect to N^1 precisely as before in respect to N, and we shall obtain, by the same steps, the same expression for the surface in terms of N^1, as was then obtained in terms of N.

To find the Centre of Gravity of a Conchoidal Solid.

Suppose the solid included between P C and Q C (fig. 3.) to be divided into an infinite number of prismatic elements by planes perpendicular to P C, and perpendicular and parallel to A z; and let $m\,r$ (fig. 5.) represent one of these elements.

The VOLUME of this element is represented by

$$\overline{n\,o} \times \overline{n\,s}$$

or by

$$\frac{1}{2}(m\,n + o\,p)\overline{n\,p . n\,s}$$

or by

$$\frac{1}{2}(mn + op)\,\overline{nr}$$

or by

$$\frac{1}{2}(um + uo)\,.\,\overline{mq}\,.\sin\Delta\Theta\,.\cos\Delta\Theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.)$$

The MOMENTUM of the element about a plane passing through A z, and perpendicular to P C, (fig. 3.) is therefore represented by

$$\frac{1}{4}(um + uo)^2\,\overline{mq}\,.\sin\Delta\Theta\,.\cos^2\Delta\Theta$$

or by

(the momentum of inertia of the plane $m\,q$) $\sin\Delta\Theta\,.\cos^2\Delta\Theta$.

Assuming then Θ to be measured from the plane $z\,y$, the momentum of the element $m\,r$ about the plane $z\,y$ is represented by

(momentum of inertia of elementary plane $m\,q$) $\sin\Theta\sin\Delta\Theta\cos^2\Delta\Theta$,

and the momentum of the same element about the plane $z\,x$ is represented by

(momentum of inertia of elementary plane $m\,q$) $\cos\Theta\sin\Delta\Theta\cos^2\Delta\Theta$.

Hence if we imagine two solids to be generated, one by the revolution of P C, without altering its dimensions, through the angle P C Q, and the other by the revolution of Q C through the same angle; and if we take I to represent the momentum of inertia of the plane P C; then will the momentum of the first solid about the plane $z\,y$, be represented by

$$I\sin\Theta\sin\Delta\Theta\cos^2\Delta\Theta,$$

and that of the second by

$$\left(I + \frac{dI}{d\Theta}\Delta\Theta + \ldots\right)\sin(\Theta + \Delta\Theta)\sin\Delta\Theta\cos^2\Delta\Theta.$$

Now the momentum of the elementary solid P C Q evidently lies between those of these elementary solids. Calling then the momentum of the whole solid, of which P C Q is an element, M_1, when estimated in respect to the plane $z\,y$, we have

$$I\sin\Theta\cos^2\Delta\Theta\sin\Delta\Theta < \frac{dM_1}{d\Theta}\Delta\Theta + \&c. \ldots < I\sin\Theta\cos^2\Delta\Theta\sin\Delta\Theta + \&c.$$

$$\therefore I\sin\Theta\cos^2\Delta\Theta < \frac{dM_1}{d\Theta}\cdot\frac{\Delta\Theta}{\sin\Delta\Theta} + \&c. \ldots < I\sin\Theta\cos^2\Delta\Theta + \&c.$$

And this is true for all values of $\Delta\Theta$.

$$\therefore \frac{dM_1}{d\Theta} = I\sin\Theta.$$

Similarly calling M_2 the moment of the whole solid about the plane $z\,x$

$$\frac{dM_2}{d\Theta} = I\cos\Theta$$

$$\therefore M_1 = \int I\sin\Theta\,d\Theta$$

$$M_2 = \int I \cos \Theta \, d\Theta$$

$\therefore$ distance of centre of gravity from plane $z\,y = \dfrac{\int I \sin \Theta \, d\Theta}{\int M \, d\Theta}$, . . (4.)

$\therefore$ distance of centre of gravity from plane $z\,x = \dfrac{\int I \cos \Theta \, d\Theta}{\int M \, d\Theta}$. . . (5.)

The generating curve has here been supposed to revolve about the axis A z, otherwise retaining its position upon it.

If we suppose P C to slide along the axis as it revolves (fig. 4.), the moment of the elementary solid P C Q about A z, and therefore the moments M_1 and M_2 of the whole solid about the planes $z\,x$ and $z\,y$ will remain unaltered.

Another dimension will however now have become necessary to determine the position of the centre of gravity; viz. its distance from a given point in the axis A z, measured along that axis.

Let V (fig. 4.) be the point where the generating curve intersects the axis A z; $\therefore$ by equation (3.) the momentum of the element $m\,r$ (fig. 5.) about a plane passing through V perpendicular to the axis A z is represented by

$$\frac{1}{2} V u \, (u\,m + u\,o) \, \overline{m\,q} \, . \sin \Delta \Theta \, . \cos \Delta \Theta \, ;$$

and assuming $V u = x$, and $u\,m$ (figs. 4 and 5.) $= y$, the momentum of the whole elementary solid generated by the revolution of P C through the angle $\Delta \Theta$ is represented by

$$\int\!\!\int x\,y\,d\,x\,d\,y \, . \sin \Delta \Theta \, . \cos \Delta \Theta .$$

And representing $\int\!\!\int x\,y\,d\,x\,d\,y$ by L, and reasoning as before, the moment of the whole solid about a plane perpendicular to A z passing through V is represented by

$$\int L \, d\Theta .$$

And if A V $= z$, the distance of the centre of gravity from A measured along the axis is represented by

$$z + \frac{\int L \, d\Theta}{\int M \, d\Theta} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

To find the Centre of Gravity of a Conchoidal Surface.

Imagine the generating curve to describe, without altering its dimensions, an angle about the axis A z (fig. 3.), such that the circular arc described on this supposition by the point P may equal the element P Q of the length of the curve or ΔS; this angle will be represented by $\frac{\Delta S}{R}$. Moreover, the moment of the elementary surface thus generated about the plane $z\,y$ will be represented by

$$\int y^2 \sin \Theta \frac{\Delta S}{R} \, d\,s,$$

where y is any ordinate of the generating curve at right angles to the axis A z, and s is taken to represent the length of the generating curve. Assuming then $\int y^2 \, d s$ or the moment of inertia of the perimeter of the curve to be represented by K, the moment of this imaginary surface about the plane $z\,y$ is represented by

$$\frac{\mathrm{K}}{\mathrm{R}} \sin \Theta \, \Delta \, \mathrm{S},$$

and, similarly, that about the plane $z\,x$ is represented by

$$\frac{\mathrm{K}}{\mathrm{R}} \cos \Theta \, \Delta \, \mathrm{S}.$$

Conceiving now a similar elementary surface to be generated by the curve Q C without changing its dimensions, the two moments of that surface will be represented by

$$\frac{\mathrm{K} + \Delta \, \mathrm{K}}{\mathrm{R} + \Delta \, \mathrm{R}} \cos (\Theta + \Delta \, \Theta) \, \Delta \, \mathrm{S}$$

and

$$\frac{\mathrm{K} + \Delta \, \mathrm{K}}{\mathrm{R} + \Delta \, \mathrm{R}} \sin (\Theta + \Delta \, \Theta) \, \Delta \, \mathrm{S}.$$

Moreover, the moment of the actual element of the conchoidal surface evidently lies between the moments of these imaginary elements; as before, therefore, the whole moments of the conchoidal surface about the planes $z\,x$ and $z\,y$, being represented by N_1 and N_2,

$$\frac{d \, \mathrm{N}_1}{d \Theta} = \frac{\mathrm{K}}{\mathrm{R}} \sin \Theta \frac{d \, \mathrm{S}}{d \Theta}$$

$$\frac{d \, \mathrm{N}_2}{d \Theta} = \frac{\mathrm{K}}{\mathrm{R}} \cos \Theta \frac{d \, \mathrm{S}}{d \Theta}.$$

Similarly, if N_3 represent the moment of the surface about a vertical plane perpendicular to the axis A z, and passing through the point V; and if x be an abscissa to any point of the generating curve measured along the axis from that point; and if H represent the integral $\int x \, y \, d \, s$, taken in respect to the whole perimeter of the generating curve; then

$$\frac{d \, \mathrm{N}_3}{d \Theta} = \frac{\mathrm{H}}{\mathrm{R}} \cdot \frac{d \, \mathrm{S}}{d \Theta}.$$

The distances of the centre of gravity from the planes $z\,y$, $z\,x$, and $x\,y$, are then respectively

$$\frac{\int \frac{\mathrm{K}}{\mathrm{R}} \sin \Theta \, d \, \mathrm{S}}{\int \frac{\mathrm{N}}{\mathrm{R}} \, d \, \mathrm{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

$$\frac{\int \frac{\mathrm{K}}{\mathrm{R}} \cos \Theta \, d \, \mathrm{S}}{\int \frac{\mathrm{N}}{\mathrm{R}} \, d \, \mathrm{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

and

$$z + \frac{\int \frac{\mathrm{H}}{\mathrm{R}} \, d \, \mathrm{S}}{\int \frac{\mathrm{N}}{\mathrm{R}} \, d \, \mathrm{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

To determine the Area of the Surface and the Centre of Gravity of a Turbinated Shell, and the Volume and Centre of Gravity of its Contained Solid.

The generating curve of a turbinated shell remains similar to itself as it revolves; the statical moment of its perimeter varies therefore as the square, and the moment of inertia of its perimeter as the cube, of any of its linear dimensions. In like manner the statical moment of the area of its generating curve varies as the cube, and the moment of inertia as the fourth power, of any of its linear dimensions.

If therefore $C_1\ C_2\ C_3\ C_4\ C_5\ C_6$ represent certain constants determined by the geometrical conditions of the generating curve,

$$N = C_1 R^2 \quad K = C_2 R^3 \quad H = C_3 R^3$$
$$M = C_4 R^3 \quad I = C_5 R^4 \quad L = C_6 R^4.$$

Therefore the surface of the shell is represented by the integral

$$C_1 \int R\, d S. \qquad (10.)$$

The co-ordinates of the centre of gravity of the surface are represented by

$$\left.\begin{array}{c} \dfrac{C_2 \int R^2 \sin \Theta\, d S}{C_1 \int R\, d S}, \quad \dfrac{C_2 \int R^2 \cos \Theta\, d S}{C_2 \int R\, d S} \\ \text{and} \\ z + \dfrac{C_3 \int R^2\, d S}{C_1 \int R\, d S}. \end{array}\right\} \qquad (11.)$$

And the volume of the contained solid is represented by

$$C_4 \int R^3\, d\, \Theta. \qquad (12.)$$

The co-ordinates of the centre of gravity of the contained solid are represented by

$$\left.\begin{array}{c} \dfrac{C_5 \int R^4 \sin \Theta\, d \Theta}{C_4 \int R^3\, d \Theta}, \quad \dfrac{C_5 \int R^4 \cos \Theta\, d \Theta}{C_4 \int R^3\, d \Theta} \\ z + \dfrac{C_6 \int R^4\, d \Theta}{C_4 \int R^3\, d \Theta}. \end{array}\right\} \qquad (13.)$$

Now it has been shown that in shells R varies according to the law of the logarithmic spiral; so that

$$R = R_0\, \varepsilon^{\Theta \cot A},$$

where R_0 is the value of R when $\Theta = 0$, and A is the constant angle which the radius vector of the spiral makes with its tangent, whether it be a plane curve or a curve of double curvature; whence it may readily be proved that

$$\frac{d S}{d \Theta} = R \operatorname{cosec} A. \qquad (14.)$$

Hence, substituting in the preceding formula, and integrating by the known rules, we obtain, for the surface of the shell, the expression

$$\frac{1}{2} C_1 R_0^2 \sec A \left(\varepsilon^{2 \Theta \cot A} - 1\right); \qquad (15.)$$

for the co-ordinates of the centre of gravity,

$$\frac{2\,C_2\,R_0}{C_1\,(\tan A+9\cot A)}\cdot\frac{(3\cot A\sin\Theta-\cos\Theta)\,\varepsilon^{3\,\Theta\cot A}+1}{\varepsilon^{2\,\Theta\cot A}-1}\quad\ldots\ldots\quad(16.)$$

$$\frac{2\,C_2\,R_0}{C_1\,(\tan A+9\cot A)}\cdot\frac{(3\cot A\cos\Theta+\sin\Theta)\,\varepsilon^{3\,\Theta\cot A}-3\cot A}{\varepsilon^{2\,\Theta\cot A}-1}\quad\ldots\quad(17.)$$

$$r_0\,(\varepsilon^{(\Theta-2n\pi)\cot A}-1)+r_0\,\varepsilon^{(\Theta-2\pi)\cot A}\cdot\left(\frac{\varepsilon^{-2n\pi\cot A}-1}{\varepsilon^{-2\pi\cot A}-1}\right)+\frac{2}{3}\left(\frac{C_3\,R_0}{C_1}\right)\left(\frac{\varepsilon^{3\Theta\cot A}-1}{\varepsilon^{2\Theta\cot A}-1}\right).\quad(18.)$$

Observing that r_0 being taken to represent the initial length of the lesser diameter* V T, of the generating curve, $r_0\,\varepsilon^{\Theta\cot A}$ will represent the length of that diameter after the generating curve has revolved through the angle Θ, and $r_0\,\varepsilon^{(\Theta-2\pi)\cot A}$ will represent the width of the next preceding whorl of the shell, measured in the direction of this diameter produced; and the sum of the widths of all the preceding whorls, supposed to be n in number, and measured in this direction, will be represented by

$$\Sigma^n{}_1\,r_0\,\varepsilon^{(\Theta-2n\pi)\cot A}.$$

Moreover, that the lesser diameter sliding along the axis, as the curve revolves through any angle, a distance precisely equal to that by which the diameter increases, it follows that the distance from the edge of the last or nth of the preceding whorls, measured in this direction, to the origin is represented by

$$r_0\,(\varepsilon^{(\Theta-2n\pi)\cot A}-1).$$

So that z_1† is represented by the formula

$$r_0\,(\varepsilon^{(\Theta-2n\pi)\cot A}-1)+\Sigma^n{}_1\,r_0\,\varepsilon^{(\Theta-2n\pi)\cot A}.$$

Integrating the formula (12.), having substituted for the value of R, we find for the VOLUME of the solid contained by the shell the expression

$$\frac{1}{3}\,C_4\,R_0^3\tan A\,(\varepsilon^{3\,\Theta\cot A}-1).\ddagger\quad\ldots\ldots\ldots\quad(19.)$$

And integrating the formula (13.), the co-ordinates of the centre of gravity of the contained solid are found to be

$$\frac{3\,C_5\,R_0}{C_4\,(\tan A+16\cot A)}\cdot\frac{(4\cot A\sin\Theta-\cos\Theta)\,\varepsilon^{4\,\Theta\cot A}+1}{\varepsilon^{3\,\Theta\cot A}-1}\quad\ldots\ldots\quad(20.)$$

$$\frac{3\,C_5\,R_0}{C_4\,(\tan A+16\cot A)}\cdot\frac{(4\cot A\cos\Theta+\sin\Theta)\,\varepsilon^{4\,\Theta\cot A}-4\cot A}{\varepsilon^{3\,\Theta\cot A}-1}\quad\ldots\quad(21.)$$

* When the whorls partially overlap one another, this diameter is to be understood to extend only across that portion of the generating curve which actually generates the chamber of the shell, and which is not interfered with by the preceding whorl. In these cases, then, it will only be a portion of what would be the shorter diameter of the generating curve, if that curve were completed.

† In the case in which the generating curve does not slide upon the axis as it revolves, $z_1=0$.

‡ In the case of turbinated shells R_0 may be considered extremely small with respect to any existing dimensions, and Θ exceedingly great, so that the formula 19. being taken to represent the whole capacity of the shell, becomes in this case $\frac{1}{3}\,C_4\,R_0^3\,\varepsilon^{3\,\Theta\cot A}$, and varies as R^3.

$$r_0\left(\varepsilon^{(\Theta-2n\pi)\cot A}-1\right)+r_0\,\varepsilon^{(\Theta-2\pi)\cot A}.\left(\frac{\varepsilon^{-2n\pi\cot A}-1}{\varepsilon^{-2\pi}-1}\right)+\frac{3}{4}\left(\frac{C_6 R_0}{C_4}\right)\left(\varepsilon^{4\Theta\cot A}-1.\right) \quad (22.)$$

To determine the Polar Equation to the surface of a Turbinated Shell.

Let m (fig. 4.) be any point in the surface of the shell, and let the equation to the curve V Q T, imagined to be in the act of generating the point m, be

$$y_1 = f(B, x_1)$$

where B is an arbitrary constant representing a linear dimension of the curve, and therefore varying according to the law of the logarithmic spiral, so that it may be represented by the formula

$$B = B_0\,\varepsilon^{\Theta\cot A},$$

where B_0 is the initial value of B. Suppose the abscissæ of the curve to be measured along the axis V T from V, so that V u and $u\,m$ are co-ordinates of m. Let A $m = \varrho$, m A $z = \Phi$, A $u = \varrho\cos\Phi = z_1 + x_1$, $u\,m = \varrho\sin\Phi = y_1$

$$\therefore \varrho\sin\Phi = f\left(B_0\,\varepsilon^{\Theta\cot A}, \varrho\cos\Phi - z_1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.)$$

or substituting for z_1 its value

$$\varrho\sin\Phi = f\left\{B_0\,\varepsilon^{\Theta\cot A}, \varrho\cos\Phi - r_0\left(\varepsilon^{(\Theta-2n\pi)\cot A}-1\right) - r_0\,\varepsilon^{(\Theta-2n\pi)\cot A}\left(\frac{\varepsilon^{-2n\pi\cot A}-1}{\varepsilon^{-2\pi\cot A}-1}\right)\right\} \quad (24.)$$

From the above may readily be determined the equation to the surface of a shell between the rectangular co-ordinates x, y, z. Observing that $\Theta - 2\,n\,\pi = \tan^{-1}\frac{x}{y}$, and substituting, we obtain

$$\left.\begin{aligned}&(x^2+y^2)^{\frac{1}{2}}\\ &= f\left\{B_0\,\varepsilon^{\left(2n\pi+\tan^{-1}\frac{x}{y}\right)\cot A}, z - r_0\left(\varepsilon^{\tan^{-1}\frac{x}{y}\cot A}-1\right) - r_0\,\varepsilon^{\tan^{-1}\frac{x}{y}\cot A}.\left(\frac{\varepsilon^{-2n\pi\cot A}-1}{\varepsilon^{-2\pi\cot A}-1}\right)\right\}\end{aligned}\right\}(25.)$$

The values of the constants $C_1\ C_2\ C_3\ C_4\ C_5\ C_6$ are dependent upon the geometrical form of the generating curve in each particular shell; the constants $R_0\ r_0$ and B_0 on its dimensions at the point where the generation of the shell is supposed to commence.

The constant A is independent of the form and dimensions of the generating curve. It depends simply upon the law of that particular logarithmic spiral which is affected by that species of shell.

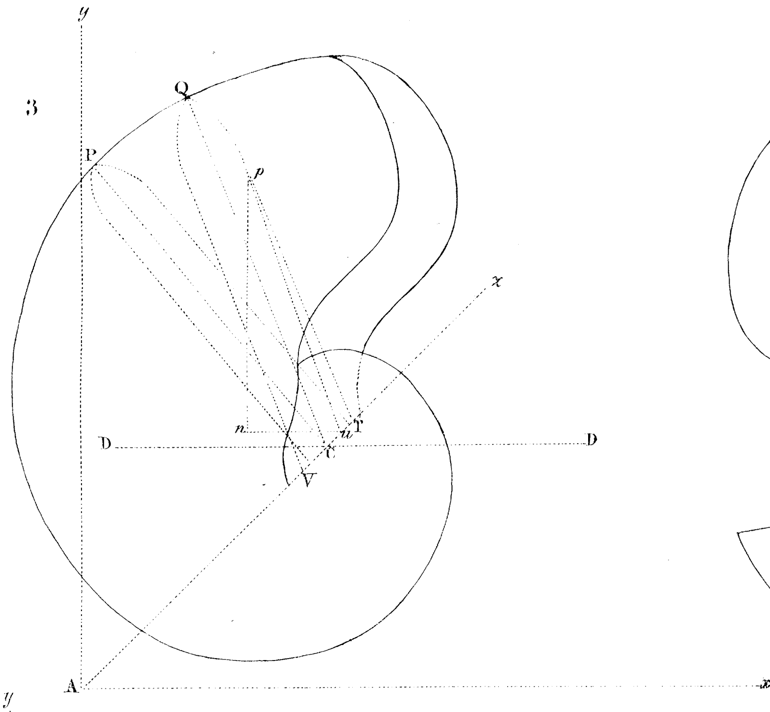
To determine the Constant Angle of the Spiral affected by any given Shell.

The common ratio of the geometrical progression according to which the widths of successive whorls increase being determined by actual admeasurement and represented by λ, we have the equation

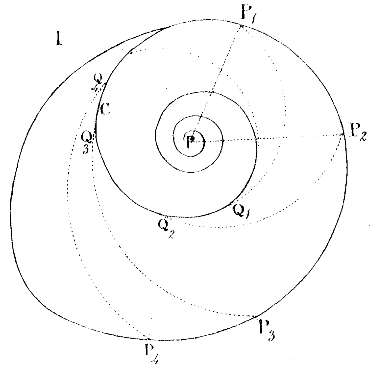
$$\varepsilon^{2\pi\cot A} = \lambda$$

$$\therefore A = \tan^{-1}\left(\frac{2\pi}{\log_\varepsilon\lambda}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26.)$$

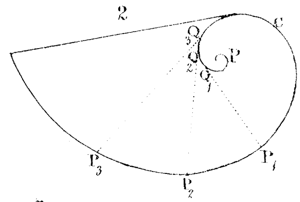
3


y
Q
P
p
x
n
u
T
D
C
D
V
A
x

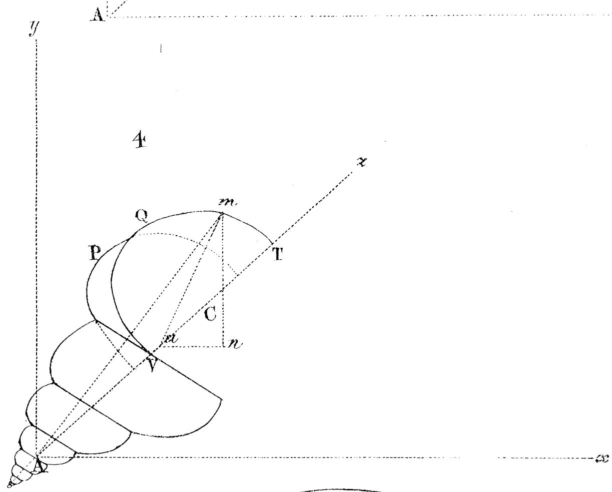
1


P1
Q4
C
Q3
P
P2
Q2
Q1
P3
P4

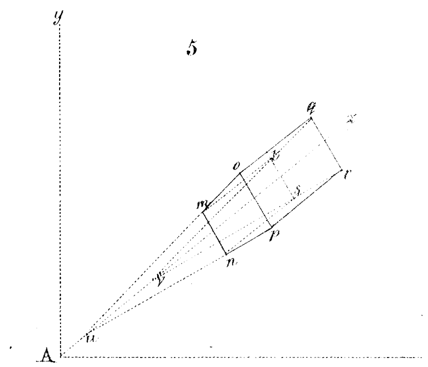
2


C
Q3
Q2
P
Q1
P1
P3
P2

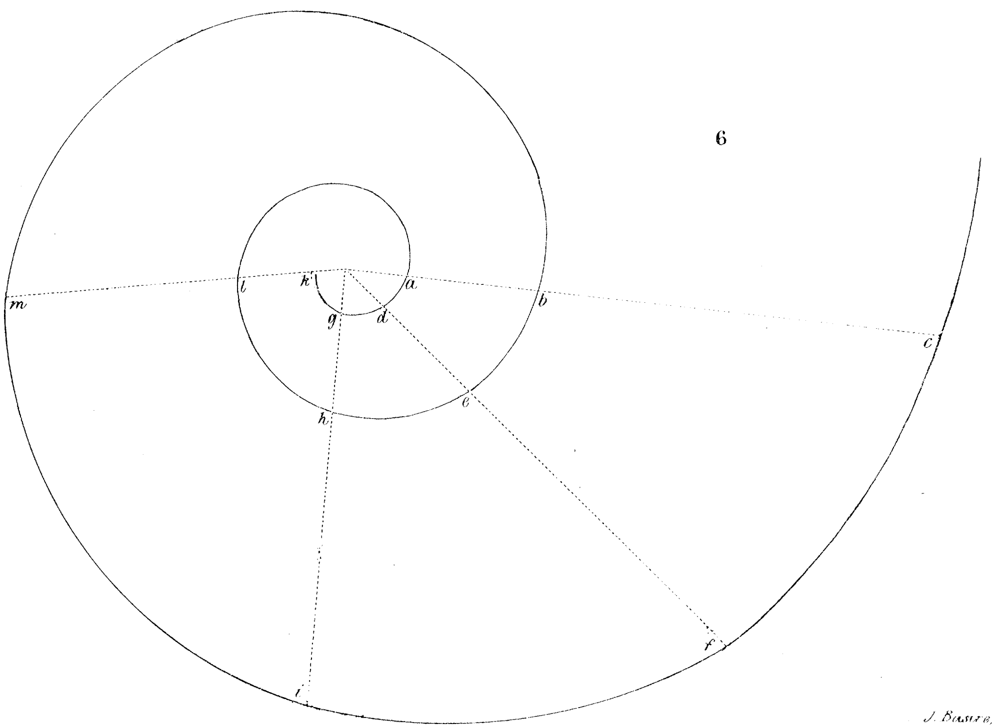
4


y
x
m
Q
P
T
C
u
n
V
A
x

5


y
q
x
o
r
m
p
n
u
A
x

6


m
l
k
a
b
g
d
c
h
e
f
i